Theorem 2. Suppose that $X$ is a discrete random variable with the probability mass function $P(X = x_i) = p_i, i = 1, 2, \cdots, r$ and support in the finite interval $[a, b]$. If $m(x)$ and $n(x)$ are both increasing (or decreasing) in $[a, b]$, then we have $m(X)$ dominates $n(X)$ by $\varepsilon$-AFSD if and only if

$$
\sum_{i=1}^{r} |n(x_i) - m(x_i)| p_i \leq \varepsilon \sum_{i=1}^{r} |n(x_i) - m(x_i)| p_i ,
$$

where $I = \{ i \mid m(x_i) < n(x_i), 1 \leq i \leq r \}$.

Proof of Theorem 2. This proof is very similar to that of Theorem 1 and the concise proof is shown as follows.

Without loss of generality, we first assume that $x_1 < x_2 < \cdots < x_r$.

(1) “If” part: Suppose $m(x)$ and $n(x)$ are both increasing in $[a, b]$. We only need to prove that

$$
\sum_{i=1}^{r} |n(x_i) - m(x_i)| p_i > \varepsilon \sum_{i=1}^{r} |n(x_i) - m(x_i)| p_i ,
$$

then there exists a utility function $u(x) \in U_{s}^{*}(\varepsilon)$, such that $E[u(m(X))] - E[u(n(X))] < 0$.

For convenience, we suppose $I = \{ s, s + 1, \cdots, t \}$. From the known conditions, we deduce that

(i) $m(x_i) \geq n(x_i)$ if $i < s$;
(ii) $m(x_i) < n(x_i)$ if $s \leq i \leq t$;
(iii) $m(x_i) \geq n(x_i)$ if $i > t$.

Define

$$
u(x) = \begin{cases} 
l_1 x, & x < m(x_i) \\
l_2 x - (l_2 - l_1) m(x_i), & m(x_i) \leq x \leq n(x_i) \\
l_1 x + (l_2 - l_1) [n(x_i) - m(x_i)], & x > n(x_i) 
\end{cases}
$$

where $l_2 > l_1 > 0$ and $\frac{l_1}{l_1 + l_2} = \varepsilon$. It is obvious that $u(x) \in U_{s}^{*}(\varepsilon)$ and (i) for any $i < s$, we have $m(x_i) \leq m(x_i)$; (ii) for any $s \leq i \leq t$, we have $m(x_i) \geq m(x_i)$ and $m(x_i) < n(x_i) \leq n(x_i)$; (iii) for any $i > t$, we have $m(x_i) \geq n(x_i) \geq n(x_i)$. Hence, we get

$$
u(m(x_i)) = \begin{cases} 
l_1 m(x_i), & i < s \\
l_2 m(x_i) - (l_2 - l_1) m(x_i), & s \leq i \leq t \\
l_1 m(x_i) + (l_2 - l_1) [n(x_i) - m(x_i)], & i > t 
\end{cases}
$$

Similarly, we have

$$
u(n(x_i)) = \begin{cases} 
l_1 n(x_i), & i < s \\
l_2 n(x_i) - (l_2 - l_1) m(x_i), & s \leq i \leq t \\
l_1 n(x_i) + (l_2 - l_1) [n(x_i) - m(x_i)], & i > t 
\end{cases}
$$

Therefore, we obtain
\[ u(m(x)) - u(n(x)) = \begin{cases} 
  l_i[m(x_i) - n(x_i)], & i < s \\
  l_s[m(x_s) - n(x_s)], & s \leq i \leq t \\
  l_i[m(x_i) - n(x_i)], & i > t 
\end{cases} \]

and

\[ E(u(m(X)) - E(u(n(X))) = \sum_{i=1}^{s} [u(m(x_i)) - u(n(x_i))]p_i \]

\[ = l_i \sum_{i=1}^{s} [m(x_i) - n(x_i)]p_i + l_s \sum_{i=s+1}^{t} [m(x_i) - n(x_i)]p_i + l_i \sum_{i=t+1}^{s} [m(x_i) - n(x_i)]p_i \]

\[ = l_i \sum_{i=1}^{s} [m(x_i) - n(x_i)] \mid p_i - (l_t + l_i) \sum_{i=s+1}^{t} [n(x_i) - m(x_i)]p_i \]

\[ = (l_t + l_i) \{ \epsilon \sum_{i=1}^{s} [m(x_i) - n(x_i)] \mid p_i - \sum_{i=s+1}^{t} [n(x_i) - m(x_i)]p_i \} \]

\[ < 0 \).

Similarly, if \( m(x) \) and \( n(x) \) are both decreasing in \([a, b]\), we only need to redefine

\[ u(x) = \begin{cases} 
  l_1x, & x < m(x_1) \\
  l_2x - (l_2 - l_1)m(x_1), & m(x_1) \leq x \leq n(x_1) \\
  l_1x + (l_2 - l_1)[n(x_1) - m(x_1)], & x > n(x_1) 
\end{cases} \]

where \( l_2 > l_1 > 0 \) and \( \frac{l_1}{l_1 + l_2} = \epsilon \). By repeating the above process, we can also deduce that

\[ E(u(m(X)) - E(u(n(X))) < 0 \).

(2) “Only if” part: Suppose \( \sum_{i=1}^{s} [m(x_i) - n(x_i)]p_i \leq \epsilon \sum_{i=1}^{s} [n(x_i) - m(x_i)] \mid p_i \), then for any \( u(x) \in U^*_\epsilon \), if \( \inf_{x \in [a, b]} u'(x) = l_1 \) and \( \sup_{x \in [a, b]} u'(x) = l_2 \), we get \( \epsilon \leq \frac{l_1}{l_1 + l_2} \). Thus, we deduce that

\[ E(u(m(X)) - E(u(n(X))) = \sum_{i=1}^{s} [u(m(x_i)) - u(n(x_i))]p_i \]

\[ = \sum_{i=1}^{s} u'(x_i)[m(x_i) - n(x_i)]p_i \quad (\text{where } x_i \text{ is among } m(x_i) \text{ and } n(x_i)) \]

\[ = \sum_{i=1}^{s} u'(x_i)[m(x_i) - n(x_i)]p_i + \sum_{i=1}^{s} u'(x_i)[m(x_i) - n(x_i)]p_i + \sum_{i=1}^{s} u'(x_i)[m(x_i) - n(x_i)]p_i \]

\[ \geq l_1 \sum_{i=1}^{s} [m(x_i) - n(x_i)]p_i + l_2 \sum_{i=1}^{s} [m(x_i) - n(x_i)]p_i + l_1 \sum_{i=1}^{s} [m(x_i) - n(x_i)]p_i \]

\[ \geq l_1 \sum_{i=1}^{s} [m(x_i) - n(x_i)] \mid p_i - (l_t + l_i) \sum_{i=s+1}^{t} [n(x_i) - m(x_i)]p_i \]

\[ = (l_t + l_i) \{ \epsilon \sum_{i=1}^{s} [m(x_i) - n(x_i)] \mid p_i - \sum_{i=s+1}^{t} [n(x_i) - m(x_i)]p_i \} \]

\[ \geq 0 \).